UDC 539.3:624.046.3 DOI: 10.31650/2786-6696-2025-11-22-32

ON PERTURBATIONS OF BUCKLING MODES OF ROD SYSTEMS CORRESPONDING TO MULTIPLE CRITICAL FORCES WHEN THE POSITION OF CONSTRAINTS CHANGES

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Abstract. The article is devoted to the study of the influence of the position of supports of rod systems containing longitudinally compressed elements on their critical forces and the corresponding forms of buckling. Many issues related to the design and operation of such systems, in particular ensuring their stability, require taking into account the features of these forms, in particular the location of their nodes, extreme points, etc. Of special complexity is the case of a multiple critical force, for which the buckling mode is not uniquely determined, since an infinitely many buckling modes correspond to a multiple critical force. In the proposed work, for the case of a concentrated deformable or absolutely rigid hinged support, it is studied how, with a small displacement of the support, two simple critical forces are formed from a multiple critical force, and two uniquely determined buckling forms are formed from the corresponding infinite set of forms. In this case, significant use is made of analytical and qualitative methods of the theory of stability of rod systems, in particular, well-known theorems on the influence of imposing constraints on their critical forces, as well as previously established relationships determining the derivatives of the critical forces with respect to the coordinates determining the positions of the moving supports. Analytical expressions are proposed that allow one to describe the buckling modes formed after a small shift of the support in one direction or another, from which, in particular, it follows that on a moving support the angles of slope of the rod axis for these forms at the same value of the support reaction are numerically equal, but opposite in direction. The conclusions of the article are demonstrated on specific examples of two-span prismatic rods compressed by a longitudinal force constant along the length. In one of them, the position of the deformable intermediate support varies with absolutely rigid end supports. In the other, the intermediate absolutely rigid support moves when one of the end supports has a finite rigidity. In both examples, at a certain value of the rigidity of the deformable support, the main critical force becomes twofold and the rod can lose stability in an infinite number of configurations. Direct calculations performed for these cases show that the shift of the intermediate support leads to the effect described in the article and confirm its results.

Keywords: stability, critical force, buckling mode, perturbation, constraint, change of position.

Introduction. Ensuring reliable operation of engineering structures requires systematic monitoring of their operational characteristics. For structures containing longitudinally compressed rods, some of the most important characteristics are critical forces and the corresponding buckling modes (forms of buckling). They are determined by the entire set of mechanical and geometric parameters of the structure, and it is very important for the designer to be able to control their behavior in connection with certain changes in these parameters. In particular, they depend on the mechanical characteristics and spatial distribution of the constraints existing in the structure. In the proposed work, the relationship of critical forces and corresponding buckling modes with the position of concentrated hinge supports that reinforce the elements of rod structures is investigated. The case of the influence of support displacement on multiple critical forces is especially studied.

Analysis of recent research. One of the ways to increase the stability of engineering structures is to increase their critical forces (hereinafter referred to as CRF), at which buckling of their compressed elements occurs, through the installation and rational placement of constraints [1 – 9]. In this case, the maximum increase in CRFs is often achieved when their multiplicity is achieved through the introduction of additional supports. To do this, the introduced supports must be located exactly at the nodes of the corresponding buckling modes (hereinafter referred to as BM). In this regard, the task of accurately determining both CRFs and BMs, to which extensive literature is devoted [1, 10, 11], is of great importance. At the same time, since the optimal placement of supports cannot always be practically realized, it is important to be able to estimate the result of deviation of the position of the introduced support from the theoretically optimal one. Such problems are the subject of perturbation theory [11 – 15] and can be studied using the general methods developed in it. However, this may leave out important information that reflects the specifics of a particular engineering problem. Therefore, it is important to consider in detail all the features and clearly identify the influence of specific perturbations on the practically important geometric and mechanical characteristics of the studied engineering objects. For a perturbation in the form of a small shift of a concentrated support, these issues in relation to a wide class of rod systems are considered in the article [7], where expressions are presented for the derivatives of the CRFs with respect to the coordinates that determine the position of the supports, both for the case of simple and multiple CRFs. These expressions make it possible to estimate, in a linear approximation, the perturbations of the CRFs as a result of the shift of supports. The question of perturbation of BMs was not considered in [7]. The BMs corresponding to simple CRFs change continuously when the position of the support changes and, to a first approximation, we can assume that they are preserved with small shifts. A more complex problem of perturbations of the BMs corresponding to multiple CRFs is considered in the proposed work.

The purpose and objectives of the study. The goal of the proposed work is to determine the buckling modes of a rod system that are formed by a small shift of a concentrated hinge support that reinforces any of its rods, provided that the corresponding critical force before the shift was multiple and an infinite number of buckling forms corresponded to it. Analytical representations of these forms are sought, allowing us to study and compare their geometric features.

Materials and methodology of the study. The study uses the main results of the mathematical theory of stability of linear elastic rod systems [1], in particular the expansion of their deformed configurations in terms of their buckling forms. A feature of the work is the systematic use of the qualitative results of this theory, describing the effect of introducing constraints and variations in their location on the critical forces of the systems under study [7].

Research results. *Preliminary results*. This section presents the main results and uses the notation of the paper [7]. We consider a system consisting of straight rods subjected to a compressive longitudinal load arbitrarily distributed along their length. The presence of areas free from compression is allowed.

Notations and assumptions. The following notations and assumptions are used below:

S – an elastic rod system, including specified elastic and rigid constraints that connect points of the system to the ground or stationary bodies.

$S^{(1)}$ – a system formed from S with the imposition of one additional constraint.

$\boldsymbol{y} = \boldsymbol{y}(M)$ – displacement (configuration, form) of the system – is a function of point M, which determines the position of point M of the deformed system (in the undeformed state $\boldsymbol{y} \equiv 0$). It is assumed that the vector $\boldsymbol{y}(M)$ is perpendicular to the axis of the undeformed rod. It is assumed that with the appropriate choice of coordinate system, the configuration is completely determined by the scalar function $y(x)$, where the coordinate x of the point M is equal to the distance measured along the axis of the corresponding rod, $y(x)$ is the numerical value of the displacement of the point M having the coordinate x.

$\boldsymbol{q} = \boldsymbol{q}(M)$ – load – a function of point M, which determines the external force applied to point M; it is assumed that the forces $\boldsymbol{q}$ applied to the rod of the system are perpendicular to the axis of the rod.

$(\boldsymbol{q}, \boldsymbol{y})$ – work of load $\boldsymbol{q} = \boldsymbol{q}(M)$ on displacement $\boldsymbol{y} = \boldsymbol{y}(M)$. If $(\boldsymbol{q}, \boldsymbol{y}) = 0$, they say that the load $\boldsymbol{q}$ is orthogonal to the form $\boldsymbol{y}$, or that the load $\boldsymbol{q}$ is applied at a generalized node of the form $\boldsymbol{y}$.

$-C\boldsymbol{y}$ – a linear operator that determines the internal forces acting on points of the system in position $\boldsymbol{y}(M)$ (including the reactions of elastic and rigid constraints belonging to the system that connect it to the ground). The "–" sign reflects the usual property of elastic structures – to generate reactions that counteract the deformation that caused them.

$N\boldsymbol{y}$ – linear operator defining external forces acting along the axis of the corresponding rod of the system. These forces form a system of couples acting on the elements of the system and arising as a result of their turn at displacement $\boldsymbol{y} = \boldsymbol{y}(M)$. On segments of the system that do not turn or bend at this displacement $(\boldsymbol{y} \equiv \mathbf{const.})$, $N\boldsymbol{y} = 0$.

$(N\boldsymbol{y}, \boldsymbol{v}) = (N\boldsymbol{v}, \boldsymbol{y})$ – work of forces of the system $N\boldsymbol{y}$, corresponding to configuration $\boldsymbol{y}$, on displacement $\boldsymbol{v}$. In particular, for a rod of length ℓ, compressed by a unit axial force constant along its length:

$$(N\boldsymbol{y}, v) = \int_0^{\ell} y'(x)\, v'(x)\, dx. \tag{1}$$

Here and below, the prime denotes the derivative with respect to the coordinate.

It is assumed that forces $N\boldsymbol{y}$ do not cause tension anywhere, but there may be segments free from compression, where $N\boldsymbol{y} = 0$. Therefore always $(N\boldsymbol{y}, \boldsymbol{y}) \geq 0$.

The equation:

$$(C - PN)\boldsymbol{y} = 0,$$

expresses the equilibrium conditions of a system under the action of systems of forces $-C\boldsymbol{y}$ and $P \cdot N\boldsymbol{y}$, where P is a parameter called the compressive force. Its nontrivial solutions $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots$, existing for a discrete set $0 \leq P_1 \leq P_2 \leq \ldots$ of values of P, constitute a set of BMs, each of which $\boldsymbol{v}_j$ corresponds to its own value P_j, called CRF,

$$(C - P_j N)\boldsymbol{v}_j = 0.$$

The forms $\boldsymbol{v}_j$ are determined up to a constant factor, which is chosen so that the normalization conditions $(N\boldsymbol{v}_j, \boldsymbol{v}_j) = 1$ are satisfied. In addition, the orthogonality relations also hold:

$$i \neq j \;\Rightarrow\; (N\boldsymbol{v}_i, \boldsymbol{v}_j) = 0.$$

These relations make the forms $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots$ convenient for constructing a basis in the space of configurations of the system S, but in the general case, in particular at the presence of segments free from compression, these forms are not sufficient to represent an arbitrary configuration as their linear combination and they should be supplemented with forms $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots$ in which all compressed segments do not turn and for which $N\boldsymbol{w}_j \equiv 0$.

If there are no internal hinges in the system and there is at least one external fastening, the configuration of the system, compressed by force P, caused by an arbitrary transverse external load $\boldsymbol{q} = \boldsymbol{q}(M)$, with completeness and an appropriate choice of normalized forms $\boldsymbol{w}_j$, can be represented in the form of an expansion:

$$\boldsymbol{y}=\sum\left(\boldsymbol{q},\boldsymbol{w}_j\right)\boldsymbol{w}_j+\sum\frac{\left(\boldsymbol{q},\boldsymbol{v}_j\right)}{P_j-P}\boldsymbol{v}_j\,. \tag{2}$$

If the load $\boldsymbol{q}=\boldsymbol{q}\left(M\right)$ is a concentrated shear force applied at a point with coordinate s and directed towards positive $y\left(x\right)$, the corresponding scalar representation of the form (2) takes the form:

$$y\left(x\right)=\sum w_j\left(s\right)w_j\left(x\right)+\sum\frac{v_j\left(s\right)}{P_j-P}v_j\left(x\right). \tag{3}$$

Perturbations of the CRFs at small shifts of a point support. If the system $S^{(1)}$ is formed from S by introducing an elastic or absolutely rigid point hinge support at a point with coordinate s, then, as was noted in [4, 7], the derivative of a simple CRF of the system $S^{(1)}$ equal to P, regardless of its number in the spectrum of CRF, is equal to:

$$P'=\frac{\partial P}{\partial s}=2\frac{Ry'\left(s\right)}{\left(N\boldsymbol{y},\boldsymbol{y}\right)}, \tag{4}$$

where R is the magnitude of the reaction of the moving support, positive when it acts in the direction opposite to the positive deflections $y\left(x\right)$.

A multiple CRF corresponds to an infinite set of BMs, the dimension of which is equal to the multiplicity of the CRF. Therefore, the expression in equation (4) loses its meaning due to uncertainty $y'\left(s\right)$. If in this case the support turns out to be in the node of each of the BMs of the system S corresponding to P, and the multiplicity of P in $S^{(1)}$ remains the same as in S, then, as follows from considerations of [7], $P'=0$, and equation (4) remains valid, because when the system $S^{(1)}$ buckles under the action of force P along any of the corresponding forms $R=0$.

If the installation of a support in the node s_0 of each of the BMs $\boldsymbol{v}_{kj}$ of the system S corresponding to its CRF equal to P_k of multiplicity r (in S), $r\geq 1$, led to the formation in $S^{(1)}$ of a new form $\boldsymbol{y}$ corresponding to the same CRF (in this case $R\neq 0$), equation (4) becomes inapplicable because in this case, the shift of the support leads to the appearance of two different simple CRFs P_b and P_a, $P_b<P_k<P_a$, the derivatives of which (one-sided) at $s=s_0$ are determined by the relations ([7]):

$$P_b'+P_a'=\frac{2Ry'\left(s_0\right)}{\left(N\boldsymbol{y},\boldsymbol{y}\right)},\quad P_b'P_a'=-\frac{R^2}{\left(N\boldsymbol{y},\boldsymbol{y}\right)}\sum_{j=1}^{r}v_{kj}'^2\left(s_0\right), \tag{5}$$

where the notation has the same meaning as in equation (4), $v_{k1}\left(x\right), v_{k2}\left(x\right), \ldots, v_{kr}\left(x\right)$ are the BMs of the system S corresponding to its CRF equal to P_k, $v_{k1}\left(s_0\right)=v_{k2}\left(s_0\right)=\ldots=v_{kr}\left(s_0\right)=0$.

Perturbations of the BMs corresponding to multiple CRFs. If, when installing a support of appropriate rigidity in a node s_0 of the form $v_k\left(x\right)$ corresponding to the CRF P_k, its multiplicity has increased, $P_{k-1}^{(1)}=P_k$, a new BM appears, which was not in S and which, according to equation (3) may be determined by the relation ([7]):

$$y\left(x\right)=-R\left[\sum w_j\left(s_0\right)w_j\left(x\right)+\sum_{j\neq k}\frac{v_j\left(s_0\right)}{P_j-P_k}v_j\left(x\right)\right], \tag{6}$$

where R is the magnitude of the reaction of the introduced support, positive when it acts in the direction opposite to the positive deflections $y\left(x\right)$. Note that in the expansion in equation (6) there are no members containing the BMs $v_k\left(x\right)$ of the system S that respond to P_k. The orthogonality condition $\left(N\boldsymbol{y},\boldsymbol{v}_k\right)=0$ follows from this.

When the support shifts from s_0 to s, the multiplicity of P_k in $S^{(1)}$ decreases and two new CRFs P_b and P_a, $P_b<P_k<P_a$ appears, the larger of which in accordance with equations (3) and (6) corresponds to the BM:

$$-R\left[\sum w_j\left(s\right)w_j\left(x\right)+\sum_{j\neq k}\frac{v_j\left(s\right)}{P_j-P_a}v_j\left(x\right)+\frac{1}{P_k-P_a}\sum_{j=1}^{r}v_{kj}\left(s\right)v_{kj}\left(x\right)\right],$$

which in the limit at $s\to s_0$ takes the form:

$$y_a\left(x\right)=y\left(x\right)+\frac{R}{P_a'}\sum_{j=1}^{r}v_{kj}'\left(s_0\right)v_{kj}\left(x\right), \tag{7}$$

where P_a' is the one-sided derivative of P_a with respect to s, calculated at $s=s_0$ from equations (5), $y\left(x\right)$ is unperturbed (before the support shift) BM of the system $S^{(1)}$, determined by equation (6).

The perturbed BM corresponding to the smaller CRF P_b is determined similarly,

$$y_b\left(x\right)=y\left(x\right)+\frac{R}{P_b'}\sum_{j=1}^{r}v_{kj}'\left(s_0\right)v_{kj}\left(x\right). \tag{8}$$

From (5), (7) and (8) the equality follows:

$$y_a'\left(s_0\right)+y_b'\left(s_0\right)=0\,.$$

which means that on a moving support the angles of slope of the rod axis for the forms $y_a\left(x\right)$ and $y_b\left(x\right)$ at the same value of the support reaction are numerically equal, but opposite in direction.

Note also that $y_a\left(x\right)$ and $y_b\left(x\right)$ from (7) and (8) satisfy the orthogonality condition $\left(N\boldsymbol{y}_a,\boldsymbol{y}_b\right)=0$.

Forms $y_a\left(x\right)$ and $y_b\left(x\right)$ give an enough accurate representation of the BMs of the system $S^{(1)}$, if s is close enough to s_0. At $s\neq s_0$ CRFs P_b and P_a are simple and their derivatives must satisfy equation (4), established for simple CRFs, i.e. following equalities must be satisfied:

$$P_a'=\left.\frac{\partial P_a}{\partial s}\right|_{s=s_0}=2\frac{Ry_a'\left(s_0\right)}{\left(N\boldsymbol{y}_a,\boldsymbol{y}_a\right)},\quad P_b'=\left.\frac{\partial P_b}{\partial s}\right|_{s=s_0}=2\frac{Ry_b'\left(s_0\right)}{\left(N\boldsymbol{y}_b,\boldsymbol{y}_b\right)}. \tag{9}$$

To verify the validity of the first of them, we note the validity of the following relations:

$$y_a'\left(s_0\right)=y'\left(s_0\right)+\frac{R}{P_a'}\sum_{j=1}^{r}v_{kj}'^2\left(s_0\right). \tag{10}$$

From equation (5) next relations follow:

$$\frac{R^2}{P_a'}\sum_{j=1}^{r}v_{kj}'^2\left(s_0\right)=-\left(N\boldsymbol{y},\boldsymbol{y}\right)P_b',\qquad Ry'\left(s_0\right)=\left(N\boldsymbol{y},\boldsymbol{y}\right)\frac{P_b'+P_a'}{2},$$

whence:

$$Ry_a'\left(s_0\right)=Ry'\left(s_0\right)+\frac{R^2}{P_a'}\sum_{j=1}^{r}v_{kj}'^2\left(s_0\right)=\left(N\boldsymbol{y},\boldsymbol{y}\right)\frac{P_b'+P_a'}{2}-\left(N\boldsymbol{y},\boldsymbol{y}\right)P_b'=\left(N\boldsymbol{y},\boldsymbol{y}\right)\frac{P_a'-P_b'}{2}. \tag{11}$$

From equation (7) taking into account orthogonality $\left(N\boldsymbol{y},\boldsymbol{v}_{kj}\right)=0$ it follows:

$$\left(N\boldsymbol{y}_a,\boldsymbol{y}_a\right)=\left(N\boldsymbol{y},\boldsymbol{y}\right)+\frac{R^2}{P_a'^2}\sum_{j=1}^{r}v_{kj}'^2\left(s_0\right)=\left(N\boldsymbol{y},\boldsymbol{y}\right)-\left(N\boldsymbol{y},\boldsymbol{y}\right)\frac{P_b'}{P_a'}=\left(N\boldsymbol{y},\boldsymbol{y}\right)\frac{P_a'-P_b'}{P_a'}. \tag{12}$$

Comparing equations (11) and (12), we get the first of the equalities:

$$\frac{2Ry_a'(s_0)}{(N\boldsymbol{y}_a,\boldsymbol{y}_a)}=P_a',\quad \frac{2Ry_b'(s_0)}{(N\boldsymbol{y}_b,\boldsymbol{y}_b)}=P_b',$$

in full accordance with equation (4). The second one is established in the same way.

We will demonstrate the results obtained using examples.

Example 1. A rectilinear prismatic rod (see Fig. 1) of length ℓ, hinged at the ends on rigid supports and compressed by a longitudinal force constant along the length, is supported by an elastic hinged support in its middle, which is the node of the 2nd BM.

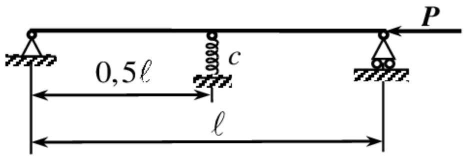


Fig. 1. A rod having a double main CRF at $c=c_{\text{cr}}=\dfrac{16\pi^2 EJ}{\ell^3}$

With a support stiffness coefficient equal to $c_{\text{cr}}=\dfrac{16\pi^2 EJ}{\ell^3}$, where EJ is the bending stiffness of the rod that is constant along the length, its main CRF becomes double and equal to the $\dfrac{4\pi^2 EJ}{\ell^2}$ – 2nd CRF of a single-span rod. It corresponds to two linearly independent BMs, defined by the equations ([10]):

$$v_2(x)=\frac{1}{\pi}\sqrt{\frac{\ell}{2}}\sin\left(\frac{2\pi x}{\ell}\right), \tag{13}$$

$$y(x)=\begin{cases}\left[\sin\left(\dfrac{2\pi x}{\ell}\right)+\dfrac{2\pi x}{\ell}\right]\cdot\dfrac{\ell^3}{16\pi^3 EJ}, \text{ if } x\le\dfrac{\ell}{2},\\ \left[\sin\left(\dfrac{2\pi(\ell-x)}{\ell}\right)+\dfrac{2\pi(\ell-x)}{\ell}\right]\cdot\dfrac{\ell^3}{16\pi^3 EJ}, \text{ if } \dfrac{\ell}{2}\le x\le\ell.\end{cases} \tag{14}$$

The form $v_2(x)$ is shown in Fig. 2 *a*), $y(x)$ – in Fig. 2 *b*). In all figures, the distance from the left end of the rod is plotted along a horizontal line and is indicated in fractions of ℓ. Vertically in Fig. 2 *b*) ordinates $y(x)$ are shown in fractions of $\dfrac{\ell^3}{\pi^3 EJ}$ at $R=1$.

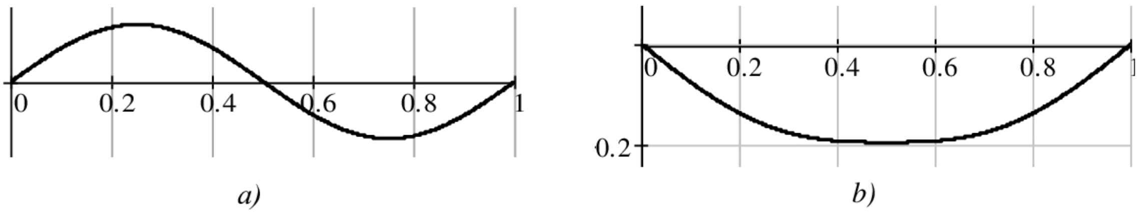


Fig. 2. BMs of the rod shown in Fig. 1, corresponding to its main double CRF

Constant coefficients in equations (13) and (14) are determined by normalization conditions. The form $v_2(x)$ satisfies the condition:

$$\left(N\boldsymbol{v}_2,\boldsymbol{v}_2\right)=\int_0^\ell v_2'^2\left(x\right)dx=1. \tag{15}$$

The form $y\left(x\right)$ corresponds to a reaction value of the intermediate support equal to 1. In this case:

$$\left(N\boldsymbol{y},\boldsymbol{y}\right)=\int_0^\ell y'^2\left(x\right)dx=\frac{3}{128}\cdot\frac{\ell^5}{\pi^4\left(EJ\right)^2}.$$

The forms $v_2\left(x\right)$ and $y\left(x\right)$ also satisfy the orthogonality condition:

$$\left(N\boldsymbol{y},\boldsymbol{v}_2\right)=\int_0^\ell y'\left(x\right)v_2'\left(x\right)dx=0.$$

Due to symmetry of $y\left(x\right)$ on the intermediate support $y'\left(\ell/2\right)=0$ (see Fig. 2 *b*)), from where, according to equation (5) equalities follow:

$$P_a'=-P_b'=\frac{\left|Rv_2'\left(\ell/2\right)\right|}{\sqrt{\left(N\boldsymbol{y},\boldsymbol{y}\right)}}=\frac{\left|v_2'\left(\ell/2\right)\right|}{\sqrt{\left(N\boldsymbol{y},\boldsymbol{y}\right)}}=\frac{16\pi^2EJ}{\sqrt{3}\ell^3}.$$

They correspond to two perturbed BMs:

$$y_a\left(x\right)=y\left(x\right)+\frac{Rv_2'\left(\ell/2\right)}{P_a'}v_2\left(x\right)=y\left(x\right)+\frac{Rv_2'\left(\ell/2\right)}{\left|Rv_2'\left(\ell/2\right)\right|}\sqrt{\left(N\boldsymbol{y},\boldsymbol{y}\right)}v_2\left(x\right)=$$

$$=y\left(x\right)-\frac{\sqrt{3}\ell^{5/2}}{8\sqrt{2}\pi^2EJ}v_2\left(x\right)=y\left(x\right)-\frac{\sqrt{3}\ell^3}{16\pi^3EJ}\sin\left(\frac{2\pi x}{\ell}\right),$$

$$y_b\left(x\right)=y\left(x\right)+\frac{\sqrt{3}\ell^3}{16\pi^3EJ}\sin\left(\frac{2\pi x}{\ell}\right).$$

The forms $y_a\left(x\right)$ and $y_b\left(x\right)$ are shown in Fig. 3 *a*) and 3 *b*) respectively.

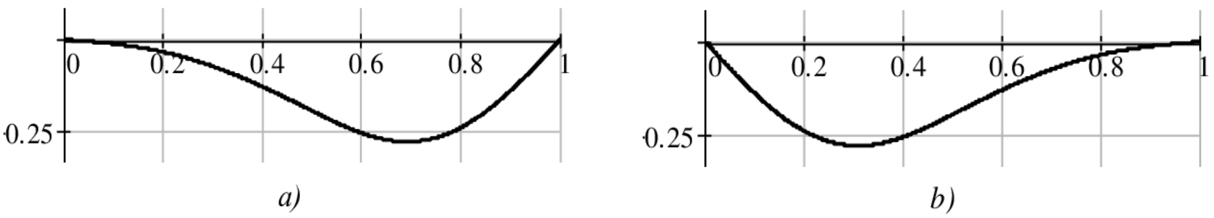


Fig. 3. Perturbed BMs of the rod shown in Fig. 1, corresponding to its main double CRF when the support shifts in the limit at $s\to s_0=\ell/2$; *a*) – $y_a\left(x\right)$, *b*) – $y_b\left(x\right)$

Example 2. A two-span rod (Fig. 4) of length ℓ, supported at the ends on hinged supports, one of which is absolutely rigid and the other deformable, is compressed by a longitudinal force constant along the length.

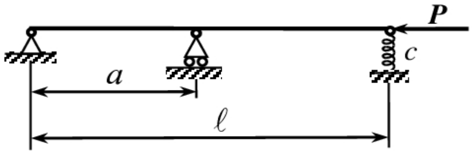


Fig. 4. A rod having a double main CRF at $c=P_2/\left(\ell-a\right)$

It has a double main CRF equal to the 2nd CRF P_2 of a single-span rod supported at the ends on absolutely rigid hinge supports, provided that the absolutely rigid intermediate support is located in the node of the 2nd BM of this single-span rod, corresponding to P_2, and the stiffness coefficient of the end deformable support is equal to $c = P_2/(\ell - a)$, where a is the coordinate of the intermediate support, equal to the distance of the node of the 2nd BM of the single-span rod from the rigid end support.

This CRF corresponds to two linearly independent BMs, one of which $v_2(x)$ is the BM of a single-span rod supported at the ends on absolutely rigid hinged supports which corresponds to its 2nd CRF equal to P_2, and the second is the semi-curved one ([2, 3, 9]) and is determined by the equalities:

$$u(x) = \begin{cases} 0, \text{ if } x \le a, \\ v_2(x) - v_2'(a)(x-a), \text{ if } a \le x \le \ell. \end{cases} \quad (16)$$

The form $u(x)$ for a prismatic rod of a constant bending stiffness is shown in Fig. 5 *a*), where the ordinates are indicated in fractions of $\frac{1}{\pi}\sqrt{\frac{\ell}{2}}$ according to the definitions of equations (13) and (16) and normalization of the form $v_2(x)$ by equation (15).

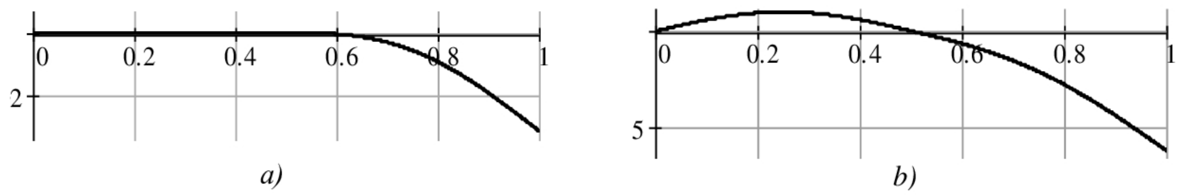


Fig. 5. BMs of the rod shown in Fig. 4 for a prismatic rod of a constant bending stiffness; *a*) – semi-curved $u(x)$, *b*) – $y(x)$ orthogonal to $v_2(x)$

Unlike the previous example, the forms $v_2(x)$ and $u(x)$ are not orthogonal. Therefore, to determine the perturbations of the CRFs and BMs, we form their linear combination:

$$y(x) = v_2(x) + \lambda u(x), \quad (17)$$

which will be orthogonal to $v_2(x)$ at

$$\lambda = -\frac{1}{\int_a^\ell v_2'^2(x)\,dx}.$$

In this case, as calculations show,

$$(N\boldsymbol{y}, \boldsymbol{y}) = \int_0^\ell y'^2(x)\,dx = \lambda^2 v_2'^2(a)(\ell - a) - (\lambda + 1).$$

The form $y(x)$ for a constant bending stiffness is shown in Fig. 5 *b*) in the same units as $u(x)$ in Fig. 5 *a*).

Since when the rod is buckled according to the form $v_2(x)$, all support reactions are equal to zero, when buckling according to the form $y(x)$ they will be the same as when buckling according to the form $\lambda u(x)$, in which the left end support is not loaded, and the two remaining reactions form a couple in which the reaction of the end support is equal to $R(\ell) = cy(\ell)$, and the reaction of the intermediate support is equal to:

$$R = R(a) = -cy(\ell) = c\lambda v_2'(a)(\ell - a) = P_2\lambda v_2'(a).$$

Moreover, since $u'(a) = 0$, from equation (17) it follows that $y'(a) = v_2'(a)$. This allows us to write down the equations (5) in the form:

$$P_b' + P_a' = \frac{2q}{Y}, \quad P_b'P_a' = -\frac{q^2}{Y}, \tag{18}$$

where $q = Rv_2'(a) = P_2\lambda v_2'^2(a)$, $Y = (N\boldsymbol{y}, \boldsymbol{y})$.

From equation (18) we find:

$$P_a' = q\frac{1-\sqrt{1+Y}}{Y} = P_2\lambda v_2'^2(a)\frac{1-\sqrt{1+Y}}{Y}, \quad P_b' = P_2\lambda v_2'^2(a)\frac{1+\sqrt{1+Y}}{Y}.$$

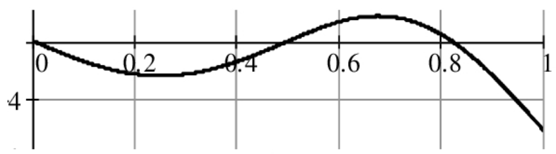
The corresponding perturbed forms according to equations (7) and (8) are determined by the equalities:

$$y_a(x) = y(x) - \left(1+\sqrt{1+Y}\right)v_2(x), \quad y_b(x) = y(x) + \left(\sqrt{1+Y} - 1\right)v_2(x). \tag{19}$$

In the case of a rod of a constant cross-section:

$$v_2(x) = \frac{1}{\pi}\sqrt{\frac{\ell}{2}}\sin\left(\frac{2\pi x}{\ell}\right), \ \lambda = -2, \quad Y = 5,$$

and perturbed BMs $y_a(x)$ and $y_b(x)$ calculated according to equation (19) take the form shown in Fig. 6 *a*) and 6 *b*) respectively.



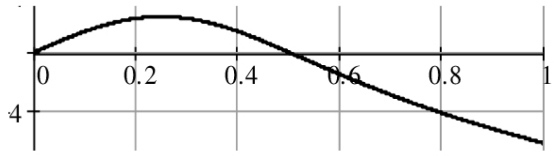


Fig. 6. Perturbed BMs of the rod shown in Fig. 4, corresponding to its main double CRF when the intermediate support is shifted in the limit at $s \to s_0 = \ell/2$; *a*) – $y_a(x)$, *b*) – $y_b(x)$

All calculations in the examples are performed on the basis of known exact analytical expressions for the influence functions of compressed prismatic rods with constant bending stiffness along the length [10].

Conclusions. The results of the article allow, in addition to the perturbations of critical forces caused by small displacement of supports, to obtain information about the appearing buckling forms of rod systems, as well as to establish some of their geometric features. This information can be used in solving various problems related to the design and operation of such systems.

The study made it possible to better understand and quantitatively estimate the influence of changes in the position of constraints on the critical forces and forms of buckling of rod systems. The use of the results presented in the article will make it possible to increase the efficiency of the design and operation of engineering structures containing elements operating under conditions of axial compression. It can be suggested that the ideas and results used in the article can be applied in the future when solving more complex problems of control and optimization of the mechanical characteristics of various engineering structures.

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ПРО ЗБУРЕННЯ ФОРМ ВТРАТИ СТІЙКОСТІ СТРИЖНЕВИХ СИСТЕМ, ЯКІ ВІДПОВІДАЮТЬ КРАТНИМ КРИТИЧНИМ СИЛАМ, ПРИ ЗМІНІ ПОЛОЖЕНЬ В'ЯЗЕЙ

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Анотація. Стаття присвячена дослідженню впливу розташування опор стрижневих систем, що містять поздовжньо стиснуті елементи, на їх критичні сили та відповідні форми втрати стійкості. Багато питань, пов'язаних з проектуванням і експлуатацією таких систем, зокрема із забезпеченням їхньої стійкості, вимагають урахування особливостей цих форм, зокрема розташування їх вузлів, точок екстремумів та ін. Особливу складність представляє випадок кратної критичної сили, для якої форма втрати стійкості не визначена однозначно, оскільки кратній критичній силі відповідає нескінченна кількість форм втрати стійкості. У запропонованій роботі для випадку зосередженої деформовної або абсолютно жорсткої шарнірної опори вивчено, як при малому зсуві опори з кратної критичної сили утворюються дві прості, а з нескінченної множини форм утворюються дві однозначно визначені форми. При цьому суттєво використовуються аналітичні та якісні методи теорії стійкості стрижневих систем, зокрема, відомі теореми про вплив накладання в'язей на їх критичні сили, а також встановлені раніше співвідношення, що визначають похідні від критичних сил по координатам, які визначають положення опор, що переміщуються. Запропоновано аналітичні вирази, які дозволяють описати знов утворені форми при малих зсувах опори в той чи інший бік, з яких, зокрема, випливає, що на опорі, що переміщується, кути нахилу осі стрижня для цих форм при одному і тому ж значенні реакції опори чисельно рівні, але протилежні за напрямком. Висновки статті продемонстровані на конкретних прикладах двопрогонових призматичних стрижнів, стиснутих постійною по довжині поздовжньою силою. В одному з них варіюється положення проміжної опори, що деформується, при абсолютно жорстких крайніх опорах. В іншому переміщується проміжна абсолютно жорстка опора, коли одна з крайніх опор має скінченну жорсткість. В обох прикладах при певному значенні жорсткості деформовної опори основна критична сила стає двократною і стрижень може втрачати стійкість по будь-якій з нескінченної множини конфігурацій. Прямі обчислення, виконані для цих випадків, показують, що зсув проміжної опори призводить до ефекту, описаного у статті, та підтверджують її результати.

Ключові слова: стійкість, критична сила, форма втрати стійкості, збурення, в’язь, зміна положення.

Стаття надійшла до редакції 3.03.2025